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ALTERNATIVE FOR AN ENCOUNTER-EVASION DIFFERENTIAL GAME WITH INTEGRAL CONSTRAINTS ON THE PLAYERS' CONTROLS

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We examine an encounter-evasion differential game with integral constraints on the players' controls. We prove that an alternative, asserting that either the position encounter problem or the position evasion problem is always solvable, is valid for this game. We indicate a position procedure for control with a guide which provides the solution to these problems. The constructions used here are modifications of the extremal construction from [1, 2], altered with due regard to the specifics of differential games with integral constraints. The present paper is also related to [3-7].

1. Let the motion of a conflict-controlled system be described by the equation

$$x^{\cdot} = f(t, x) + B(t, x)\, u + C(t, x)\, v \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are the controls of the first and second players, respectively; $f(t, x)$, $B(t, x)$ and $C(t, x)$ are continuous vector-valued and matrix-valued functions which satisfy a Lipschitz condition in variable x in each bounded region. We note that under the assumptions made relative to the right-hand side of (1.1), a unique solution of the Cauchy problem for Eq. (1.1) with initial condition $x(t_0) = x_0$ exists for any choice of summable functions $u(t)$ and $v(t)$.

We assume that the realizations $u(t)$ and $v(t)$ of the players' controls satisfy the constraints

$$I_u(t_0, \infty) \leqslant \mu(t_0), \quad I_v(t_0, \infty) \leqslant \nu(t_0) \tag{1.2}$$

Here and subsequently

$$I_u(a, b) = \left(\int_a^b \|u(t)\|^p dt\right)^{1/p}, \quad I_v(a, b) = \left(\int_a^b \|v(t)\|^q dt\right)^{1/p}$$

the symbol $\|w\|$ denotes the Euclidean norm of vector w. Summable realizations of the controls $u(t)$, $v(t)$ $(t \geqslant t_0)$, satisfying conditions (1.2), are said to be addmissible. We assume that system (1.1) and constraints (1.2) are such that for any admissible realizations $u(t)$ and $v(t)$ $(t \geqslant t_0)$ the Cauchy problem mentioned can be continued up to any instant $t = T$; a bounded region $G(T)$ exists in which the solutions examined on the interval $[t_0, T]$ remain for any choice of admissible realizations.

The change in the constraints $\mu(t)$ and $\nu(t)$ is determined by the players' control resources spent during the game, i.e.

$$\mu(t_2) = \mu(t_1) - I_u(t_1, t_2), \quad \nu(t_2) = \nu(t_1) - I_v(t_1, t_2)$$

The vector $z = (t, \mu, \nu, x)$ is called the game's position. Note that $\mu \geqslant 0, \nu \geqslant 0$

everywhere in the following. Certain closed sets M^* and N^* are specified in the space of vectors (t, x).

The encounter problem facing the first player consists in choosing the control u so as to ensure that the point $(t, x(t))$ hits onto the set M^* and that the phase constraint $(t, x(t)) \in N^*$ is fulfilled for $t_0 \leqslant t \leqslant \tau$, where τ is the instant at which the condition $(t, x(t)) \in M^*$ first is realized. It is assumed here that the first player knows the current position of the game, namely, the vector $z(t)$.

The evasion problem facing the second player consists in choosing the control v so as to ensure either that the point $(t, x(t))$ evades contact with set M^* or that the phase constraint $(t, x(t)) \in N^*$ is violated before the condition $(t, x(t)) \in M^*$ is realized. Here information on the game position realized is available to the second player.

The encounter and evasion problems are examined here for the case when the sets M^* and N^* are contained in a closed region where $t \leqslant \vartheta$ ($\vartheta > t_0$ is the instant bounding the game's duration). We note that the pursuit problem solutions and several other differential games are reduced to investigation of the encounter-evasion game which is composed of these two problems (cf. [1, 2]). In the present paper we deduce a scheme for proving an alternative which asserts that either the evasion problem or the encounter problem is always solvable in the encounter-evasion game.

2. We describe a position procedure of control with a guide, which provides the solution of the encounter problem. The constructions proposed below are analogous to the constructions in [1, 2]. We introduce auxiliary concepts and notation.

Let $z_* = (t_*, \mu_*, \nu_*, x_*)$ be some game position, $v(t)$ $(t \geqslant t_*)$ be an admissible realization of the second player's control, a function satisfying condition $I_v(t_* \ \infty) \leqslant \nu_*$. The symbol $G^{(u)}(z_*, v(\cdot))$ denotes a set of points $z = (t, \mu(t), \nu(t), x(t))$ of the form

$$t \geqslant t_* 0 \leqslant \mu^p(t) \leqslant \mu_*{}^p - I_u(t_*, t)^p, \qquad \nu^q(t) = \nu_*{}^q - I_v(t_*, t)^q$$

$$x(t) = x_* + \int_{t_*}^{t} [f(\sigma, x(\sigma)) + B(\sigma, x(\sigma))u(\sigma) + C(\sigma, x(\sigma))v(\sigma)]\, d\sigma$$

where $u(\sigma)$ $(\sigma \geqslant t_*)$ are all possible summable functions satisfying the condition $I_u(t_*, \infty) \leqslant \mu_*$. Let D be some set in the $(n+3)$-dimensional space of vectors $z = (t, \mu, \nu, x)$. The symbol $D_{[t_*, t^*]}$ denotes the part of this set lying between the hyperplanes $t = t_*$ and $t = t^*$, i.e.

$$D_{[t_*, t^*]} = \{z : z = (t, \mu, \nu, x) \in D, \ t_* \leqslant t \leqslant t^*\}$$

The section of set D by the hyperplane $t = t_*$ is denoted by the symbol D_{t_*}, i.e.

$$D_{t_*} = \{z: z = (t_*, \mu, \nu, x) \in D\}$$

The symbols M and N denote sets in the space of positions z, defined by the relations

$$M = \{z = (t, \mu, \nu, x): (t, x) \in M^*, \ \mu \geqslant 0, \ \nu \geqslant 0\}$$
$$N = \{z = (t, \mu, \nu, x): (t, x) \in N^*, \ \mu \geqslant 0, \ \nu \geqslant 0\}$$

Definition 2.1. Let a certain set $W^{(u)}$ be given in the position space. This set is said to be a u-stable bridge if $W^{(u)} \subset N$, $W_\vartheta^{(u)} \subset M$ and the following condition is satisfied: the relation

$$G^{(u)}(z_*, v(\cdot)) \cap (W_{t^*}^{(u)} \cup M_{[t_*, t^*]}) \neq \phi$$

is valid for any point $z_* = (t_*, \mu_*, \nu_*, x_*) \in W^{(u)}$, for any number $t^* > t_*$ and for any realization $v(t)$ $(t \geqslant t_*)$ admissible for position z_*.

We note that in this definition we have allowed the case $W_\vartheta^{(u)} = \varnothing$ and have not assumed the closedness of the set $W^{(u)}$; however, it can be verified that for any u-stable bridge $W^{(u)}$ its closure also is a u-stable bridge.

We introduce into consideration the functions

$$u_*(z, z^*, \delta) = \begin{cases} -\dfrac{b}{\|b\|}(\mu^p - \mu^{*p})^{1/p}\delta^{-1/p} & \text{for } \mu - \mu^* > 0, \ \|b\| \neq 0 \\ 0 \quad \text{for } \mu - \mu^* \leqslant 0 \quad \text{or} \quad \|b\| = 0 \end{cases} \tag{2.1}$$

$$v^*(z, z^*, \delta) = \begin{cases} \dfrac{c}{\|c\|}(\nu^{*q} - \nu^q)^{1/q}\delta^{-1/q} & \text{for } \nu^* - \nu > 0, \ \|c\| \neq 0 \\ 0 \quad \text{for } \nu^* - \nu \leqslant 0 \quad \text{or} \quad \|c\| = 0 \end{cases} \tag{2.2}$$

$$u^*(z, z^*, \delta) = \begin{cases} \dfrac{b}{\|b\|}(\mu^{*p} - \mu^p)^{1/p}\delta^{-1/p} & \text{for } \mu^* - \mu > 0, \ \|b\| \neq 0 \\ 0 \quad \text{for } \mu^* - \mu \leqslant 0 \quad \text{or} \quad \|b\| = 0 \end{cases} \tag{2.3}$$

$$v_*(z, z^*, \delta) = \begin{cases} -\dfrac{c}{\|c\|}(\nu^{*q} - \nu^q)^{1/q}\delta^{-1/q} & \text{for } \nu - \nu^* > 0, \ \|c\| \neq 0 \\ 0 \quad \text{for } \nu - \nu^* \leqslant 0 \quad \text{or} \quad \|c\| = 0 \end{cases} \tag{2.4}$$

Here (the primes denote transposition)

$$z = (t_*, \mu, \nu, x), \ z^* = (t_*, \mu^*, \nu^*, x^*), \delta > 0, \ b' = (x - x^*)' \times B(t_*, x), \ c' = (x - x_*)' C(t_*, x)$$

The functions u_* in (2.1) and v^* in (2.2) are used here to construct for the first player a procedure of control with a guide; the functions u^* in (2.3) and v_* in (2.4) are applied below to determine for the second player a procedure of control with a guide.

Note that when $\mu - \mu^* > 0$, $\|b\| \neq 0$ and $\nu - \nu^* > 0$, $\|c\| \neq 0$, the functions $u_*(t) = u_*(z, z^*, \delta) = \text{const}$ and $v_*(t) = v_*(z, z^*, \delta) = \text{const}$ $(t_* \leqslant t \leqslant t_* + \delta)$ provides a minimum to the functionals

$$(x - x^*)' B(t_*, x) \int_{t_*}^{t_*+\delta} u(t)\,dt, \quad (x - x^*)' C(t_*, x) \int_{t_*}^{t_*+\delta} v(t)\,dt \tag{2.5}$$

which are examined on the set of functions $u(t)$ and $v(t)$ $(t_* \leqslant t \leqslant t_* + \delta)$ satisfying the conditions

$$I_u(t_*, t_* + \delta) \leqslant (\mu^p - \mu^{*p})^{1/p}, \quad I_v(t_*, t_* + \delta) \leqslant (\nu^q - \nu^{*q})^{1/q} \tag{2.6}$$

When $\mu^* - \mu > 0$, $\|b\| \neq 0$ and $\nu^* - \nu > 0$, $\|c\| \neq 0$ the functions $u^*(t) = u^*(z, z^*, \delta)$, $v^*(t) = v^*(z, z^*, \delta)$ $(t_* \leqslant t \leqslant t_* + \delta)$ provide a maximum to functionals (2.5) which are examined on the set of functions satisfying constraints (2.6) wherein we interchange the positions of μ and μ^*, ν and ν^*.

Suppose we are given an initial position $z_* = (t_*, \mu_*, \nu_*, x_*)$ and the bridge $W^{(u)}$ and let the set $W_{t_*}^{(u)}$ be nonempty and closed. For this bridge let us define a guide-control procedure for the first player. We select the point

$$z^*(t_*) = (t_*, \mu^*, \nu^*, x^*) \in W_{t_*}^{(u)}$$

nearest to point $z(t_*) = z_*$. This point $z^*(t_*)$ is the position of the guide at the initial instant $t = t_*$. We choose a certain covering Δ of the interval $[t_*, \vartheta)$ by a system of semi-intervals $[\tau_i, \tau_{i+1})$ of equal length $\delta = \tau_{i+1} - \tau_i$ $(i = 0, 1, \ldots, n, \tau_0 = t_*, \tau_N = \vartheta)$. We assume that on the first interval $[\tau_0, \tau_1)$ the motion of system (1.1) is generated by the first player's continuous control

$$u^{(0)}(t) = u_*(z(t_*),\ z^*(t_*),\ \delta)\ \ (\tau_0 \leqslant t \leqslant \tau_1)$$

in pair with some admissible realization $v(t)$ $(t \geqslant t_0)$ of the second player's control. The choice of these controls determines the game position $z(\tau_1) = (\tau_1, \mu(\tau_1), \nu(\tau_1), x(\tau_1))$ realized at instant $t = \tau_1$.

Let

$$v^{(0)}(t) = v^*(z(t_*), z^*(t_*))\ \ (\tau_0 \leqslant t < \tau_1)$$

We choose the guide's position $z^*(\tau_1)$ at instant $t = \tau_1$ from the condition

$$z^*(\tau_1) \in W_{\tau_1}^{(u)} \cap G^{(u)}(z^*(t_*), v^{(0)}(\cdot))$$

assuming that this intersection is nonempty. Next, on the succeeding interval $[\tau_1, \tau_2)$ we determine the first player's control in system (1.1) by the relation

$$u^{(1)}(t) = u_*(z(\tau_1),\ z^*(\tau_1), \delta)\ \ (\tau_1 \leqslant t < \tau_2)$$

As a result of choosing this control and some control $v(t)$ $(t \geqslant \tau_1)$ of the second player at instant $t = \tau_2$, the game position $z(\tau_2)$ is realized. We select

$$v^{(1)}(t) = v^*(z(\tau_1),\ z^*(\tau_1),\ \delta)\ \ (\tau_1 \leqslant t < \tau_2)$$

and we determine the guide's position at instant $t = \tau_2$ from the condition

$$z^*(\tau_2) \in W_{\tau_2}^{(u)} \cap G^{(u)}(z^*(\tau_1),\ v^{(1)}(\cdot))$$

assuming once again that this intersection is not empty. If the condition

$$W_{\tau_{i+1}}^{(u)} \cap G^{(u)}(z^*(\tau_i),\ v^{(i)}(\cdot)) \neq \phi$$

is also satisfied on the succeeding intervals $[\tau_i, \tau_{i+1})$, then the indicated control procedure is implemented up to the instant $t = \vartheta$.

Let us now consider the case when this condition is not satisfied. Let τ_j be the instant when first

$$W_{\tau_j}^{(u)} \cap G^{(u)}(z^*(\tau_{j-1}), v^{(j-1)}(\cdot)) = \phi$$

Then from the condition $z^*(\tau_{j-1}) \in W^{(u)}$ and from the definition of a u-stable bridge it follows that

$$M_{[\tau_{j-1}, \tau_j]} \cap G^{(u)}(z^*(\tau_{j-1}),\quad v^{(j-1)}(\cdot)) \neq \phi$$

i.e. an instant $\tau^* \in [\tau_{j-1}, \tau_j]$, exists at which the guide's position can be determined from the condition

$$z^*(\tau^*) \in M_{\tau^*} \cap G(z^*(\tau_{j-1}),\ v^{(j-1)}(\cdot))$$

Then, we assume that an arbitrary point $z^*(\tau_j) \in G_{\tau_j}(z^*(\tau^*),\ v^{(j-1)}(\cdot))$ has been chosen as the guide's position $z^*(\tau_j)$ at the instant $t = \tau_j$. Further, we determine the controls $u^{(i)}(t)$ and $v^{(i)}(t)$ $(\tau_i \leqslant t < \tau_{i+1},\ j \leqslant i \leqslant n-1)$ by the relations

$$u^{(i)}(t) = u_*(z(\tau_i),\ z^*(\tau_i),\ \delta),\ v^{(i)}(t) = v^*(z(\tau_i),\ z^*(\tau_i),\ \delta)$$

and we select the guide's position $z^*(\tau_{i+1})$ arbitrarily from the set $G_{\tau_{i+1}}^{(u)}$ $(z^*(\tau_i),$

$v^{(i)}(\cdot))$.

We note that the first player's control

$$u_\Delta(t) = u_*(z(\tau_i), z^*(\tau_i), \delta)\ (\tau_i \leqslant t \leqslant \tau_{i+1},\ i = 0, \ldots, n-1) \quad (2.7)$$

constructed here does not violate the constraint $I_{u_\Delta}(t_*, \vartheta) \leqslant \mu_* = \mu(t_*)$. The motion of system (1.1), realized by the choice of the guide-control $u_\Delta(t)$ of (2.7) in pair with some second player's control $v(t)$ $(t \geqslant t_*)$, is called approximated and is denoted by the symbol $x(t;\ t_*, u_\Delta(\cdot), v(\cdot))$ $(t_0 \leqslant t \leqslant \vartheta)$.

Definition 2.2. The function $x(t)$ $(t_* \leqslant t \leqslant \vartheta)$ is called the motion of system (1.1), generated by the first player's guide-control procedure, if a sequence of approximated motions

$$x_k(t) = x(t;\ z_*^{(k)}, u_{\Delta_k}(\cdot), v(\cdot))\ (t_* \leqslant t \leqslant \vartheta,\ k = 1, 2, \ldots)$$

exists satisfying the conditions

$$\delta^{(k)} = (\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0, \quad z_*^{(k)} = (t_*, \mu_*^{(k)}, \nu_*^{(k)}, x_*^{(k)}) \to z_* = (t_*, \mu_*, \quad (2.8)$$
$$\nu_*, x_*), \ \max_{t_* \leqslant t \leqslant \vartheta} \|x(t) - x_k(t)\| \to 0 \quad \text{as} \quad k \to \infty$$

The motions introduced here are denoted by the symbol $x(t;\ z_*, W^{(u)})$ which indicates the initial position of the u-stable bridge for which the guide-control procedure is determined. We note that the existence of the motions $x(t;\ z_*, W^{(u)})$ $(t_* \leqslant t \leqslant \vartheta)$ follows from the Arzelà's theorem since the approximated motions are equibounded and equicontinuous. The fundamental property of a guide-control procedure is formulated in the following way.

Theorem 2.1. If there exists a u-stable bridge $W^{(u)}$ containing the initial game position z_*, then for any motion $x(t;\ z_*, W^{(u)})$ there exists an instant $\tau \leqslant \vartheta$ when first the point $(t, x(t;\ z_*, W^{(u)}))$ hits the set M^*, and the conditions

$$(t, x(t;\ z_*, W^{(u)})) \in N^* \quad \text{for} \quad t_* \leqslant t \leqslant \vartheta$$

are satisfied.

This theorem's proof is based on estimating the distance between the motion of the original system (1.1) and the motion of the guide, and it is ascertained that the choice of controls (2.3), (2.4) ensures the proximity of these motions when a sufficiently small step is chosen.

3. We consider the solution of the evasion problem, relying on a position procedure with a guide, and we prove an alternative for the encounter-evasion game.

Let $z_* = (t_*, \mu_*, \nu_*, x_*)$ be some game position, $u(\sigma)$ $(\sigma \geqslant t_*)$ be some realization of the first player's control, admissible for this position, i.e. $u(\sigma)$ $(\sigma \geqslant t_*)$ is a summable function satisfying the condition $I_u(t_*, \infty) \leqslant \mu_*$. The symbol $G^{(v)}(z_*, u(\cdot))$ denotes the set of points $z = (t, \mu(t), \nu(t), x(t))$ of the form

$$t \geqslant t_*, \quad \mu^p(t) = \mu_*{}^p - I_u(t_*, t)^p, \quad 0 \leqslant \nu^q(t) \leqslant \nu_*{}^q - I_v(t_*, t)^q$$

$$x(t) = x_* + \int_{t_*}^{t} [f(\sigma, x(\sigma)) + B(\sigma, x(\sigma))u(\sigma) + C(\sigma, x(\sigma))v(\sigma)]\, d\sigma$$

where $v(\sigma)$ $(\sigma \geqslant t_*)$ are all possible summable functions satisfying the condition

$I_v(t_*, \infty) \leqslant \nu_*$. Let

$$\begin{aligned} G &= \{z = (t, \mu, \nu, x): (t, x) \in G^*, \ \mu \geqslant 0, \ \nu \geqslant 0\} \\ H &= \{z = (t, \mu, \nu, x): (t, x) \in H^*, \ \mu \geqslant 0, \ \nu \geqslant 0\} \end{aligned} \tag{3.1}$$

where G^* and H^* are certain closed sets in the space of vectors (t, x), satisfying the conditions

$$G^* \cap M^* = \phi, \ H^* \cap N^* = \phi \tag{3.2}$$

Definition 3.1. Let a certain set $W^{(v)}$ be specified in the space of vectors z. This set is called a v-stable bridge if $W^{(v)} \subset G$ and if the relation

$$G^{(v)}(z_*, u(\cdot)) \cap (W_{t*}^{(u)} \cup H_{[t_*, t^*]}) \neq \phi \tag{3.3}$$

where H and G are certain sets (3.1), (3.2), is valid for any point $z_* = (t_*, \mu_*, \nu_*, x_*) \in W^{(v)}$, for any number $t^* \in (t_*, \vartheta]$, and for any control $u(t)$ $(t \geqslant t_*)$ admissible for position z_*.

For a v-stable bridge $W^{(v)}$ let us determine the second-player's guide-control procedure. Assuming that $v_*(z, z^*, \delta)$ is defined by relation (2.4), we form the second player's control in system (1.1) in the following way:

$$v_\Delta(t) = v_*(z(\tau_i), z^*(\tau_i), \delta) \ (\tau_i \leqslant t < \tau_{i+1}; \ i = 0, \ldots, n-1) \tag{3.4}$$

Here $z(\tau_i)$ is the game position realized at instant $t = \tau_i$ for the choice of control $v_\Delta(t)$ of (3.4) in pair with some admissible control $u(t)$ $(t_* \leqslant t < \tau_i)$ of the first player, $z^*(\tau_i)$ is the guide's position at the instant $t = \tau_i$.

In the determination of the guide's position we use the equations

$$\begin{gathered} u^{(i)}(t) = u^*(z(\tau_i), \ z^*(\tau_i), \ \delta) \\ (\tau_i \leqslant t < \tau_{i+1} = \tau_i + \delta, \ i = 0, 1, \ldots, n-1) \end{gathered}$$

where the function $u^*(z, z^*, \delta)$ is defined by relations (2.3). As the guide's initial position $z^*(t_*)$ we select the point of set $W_{t_*}^{(v)}$ nearest to the initial game position $z = (t_*, \mu_*, \nu_*, x_*)$ (presuming that $W_{t_*}^{(v)}$ is closed and nonempty). Next, we determine the guide's positions $z^*(\tau_i)$ successively from the condition

$$z^*(\tau_i) \in G^{(v)}(z^*(\tau_{i-1}), \ u^{(i-1)}(\cdot)) \cap W_{\tau_i}^{(v)}$$

either up to the instant $\tau_N = \vartheta$, if all these intersections are not empty, or up to the instant τ_j for which this intersection proves to be empty. We determine the position $z^*(\tau_j)$ at the instant $t = \tau_j$ from the condition

$$\begin{gathered} z^*(\tau_j) \in G_{\tau_i}^{(v)}(z^*(\tau^*), \quad u^{(j-1)}(\cdot)), \quad z^*(\tau^*) \in \\ G_{\tau_*}^{(v)}(z^*(\tau_{j-1}), \ u^{(j-1)}(\cdot)) \cap H, \quad \tau_{j-1} \leqslant \tau^* \leqslant \tau_j \end{gathered}$$

The existence of such a point $z^*(\tau^*)$ follows from the condition $z^*(\tau_{j-1}) \in W_{\tau_{j-1}}^{(v)}$ and from the definition of the v-stable bridge $W^{(v)}$. Next, for $j \leqslant i \leqslant n$, we select arbitrary points of the sets $G_{\tau_i}^{(v)}(z^*(\tau_{i-1}), \ u^{(i-1)}(\cdot))$ as the guide's positions $z^*(\tau_i)$. We note that the control $v_\Delta(t)$ $(t_* \leqslant t \leqslant \vartheta)$ of (3.4) does not violate the constraint $I_{v_\Delta}(t_*, \vartheta) \leqslant \nu_* = \nu(t_*)$. The motion of system (1.1), realized by the choice of the second player's guide-control $v_\Delta(t)$ $(t_* \leqslant t \leqslant \vartheta)$ of (3.4) in pair with some first player's admissible control $u(t)$, $(t_* \leqslant t \leqslant \vartheta)$, is called approximated and is denoted by the symbol $x(t; z_*, u(\cdot), v_\Delta(\cdot))$ $(t_* \leqslant t \leqslant \vartheta)$.

Definition 3.2. The function $x(t)$ $(t_* \leqslant t \leqslant \vartheta)$ is called the motion of system (1.1), generated by the second player's guide-control procedure, if a sequence of approximated motions

$$x_k(t) = x(t;\ z_*^{(k)},\ u^{(k)}(\cdot),\ v_{\Delta_k}(\cdot)) \quad (t_* \leqslant t \leqslant \vartheta,\ k = 1, 2, \ldots)$$

exists satisfying conditions (2.8).

The motions defined here are denoted by the symbol $x(t;\ z_*,\ W^{(v)})$ $(t_* \leqslant t \leqslant \vartheta)$ The following statement is valid.

Theorem 3.1. If there exists a v-stable bridge $W^{(v)}$ containing the initial game position z_*, then for any motion $x(t;\ z_*,\ W^{(v)})$ $(t_* \leqslant t \leqslant \vartheta)$ the point $(t,\ x(t;\ z_*,\ W^{(v)}))$ remains in region G^* either up to the instant $t = \vartheta$ or up to the instant τ^* when first it hits the set H^*, i.e. all motions $x(t;\ z_*,\ W^{(v)})$ evade contact with set M^*.

Let us consider certain properties of the solution of the evasion problem, usable in the proof of the alternative for the encounter-evasion game. Let

$$z_*^{(k)} = (t_*,\ \mu_*^{(k)},\ \nu_*^{(k)},\ x_*^{(k)}) \to z_* = (t_*,\ \mu_*,\ \nu_*,\ x_*)$$
$$x^{(k)}(t) = x(t;\ z_*^{(k)},\ W^{(v)}) \to x_*(t)\ (t_* \leqslant t \leqslant \vartheta) \quad \text{as} \quad k \to \infty$$

then the function $x_*(t)$ is one of the motions $x(t;\ z_*,\ W^{(v)})$, this follows directly from the definition of the latter. From this property of the semicontinuous dependence of the motions $x(t;\ z_*,\ W^{(v)})$ $(t_* \leqslant t \leqslant \vartheta)$ on position z_*, the following statement results.

Lemma 3.1. If position $z_* = (t_*,\ \mu_*,\ \nu_*,\ x_*)$ belongs to some v-stable bridge $W^{(v)}$, then there exist an ε-neighborhood of this position

$$S(z_*,\ \varepsilon) = \{z = (t_*,\ \mu,\ \nu,\ x)\colon\ |\mu - \mu_*| \leqslant \varepsilon,\ \ \mu \geqslant 0,\ \ |\nu - \nu^*| \leqslant \varepsilon,\ \nu \geqslant 0,\ \|x - x_*\| \leqslant \varepsilon\} \tag{3.5}$$

and closed sets G_ε^*, H_ε^*, satisfying conditions (3.2), such that for any motion $x(t;\ z_\varepsilon,\ W^{(v)})$ $(t_* \leqslant t \leqslant \vartheta)$, where $z_\varepsilon \in S(z_*,\ \varepsilon)$, the point $(t,\ x(t;\ z_\varepsilon,\ W^{(v)}))$ remains in region G_ε^* either for $t_* \leqslant t \leqslant \vartheta$ or up to the instant of its first contact with set H_ε^* (i.e. all motions $x(t;\ z_\varepsilon,\ W^{(v)})$ evade contact with M^*).

Using this statement we can prove the following assertion.

Lemma 3.2. If position $z_* = (t_*,\ \mu_*,\ \nu_*,\ x_*)$ belongs to some v-stable bridge $W^{(v)}$ then there exists a v-stable bridge $W_*^{(v)}$ containing not only the point z_* but also some ε-neighborhood $S(z_*,\ \varepsilon)$ $(\varepsilon > 0)$ (3.5) of this point.

Let $W_0^{(u)}$ be the complement of set $W_\Sigma^{(v)}$ which is the union of all v-stable bridges, i.e.

$$W_\Sigma^{(v)} = \bigcup W^{(v)}$$

The following statement is valid.

Theorem 3.2. Set $W_0^{(u)}$ is a u-stable bridge.

We indicate the highlights of the proof of this theorem. From the fact that every set H of form (3.1), (3.2) is a v-stable bridge it follows that the union of such sets, being also the complement of set N, is contained in the set $W_\Sigma^{(v)}$. Consequently, the inclusion

$$W_0^{(u)} \subset N$$

is true. Let us show that the relation $W_{0\vartheta}^{(u)} \subset M$ is valid. In fact, every set G satisfying the condition $G \cap M_\vartheta = \varnothing$ is a v-stable bridge. Therefore, the union of all such sets G, forming the complement to set M_ϑ, belongs to set $W^{(v)}_{\Sigma,\vartheta}$, i.e.

$$C(M_\vartheta) \subset W^{(v)}_{\Sigma,\vartheta}, \qquad W^{(u)}_{0\vartheta} = C\,(W^{(v)}_{\Sigma,\vartheta}) \subset M_\vartheta$$

Q.E.D.

It remains to verify the fulfillment of the relation

$$G^{(u)}(z_*, v(\cdot)) \cap (W^{(u)}_{0t_*} \cup M_{[t_*,t^*]}) \neq \phi$$

We assume the contrary. Let there exist a point $z_* \in W^{(u)}_{0t_*}$, a number $t^* \in (t_*, \vartheta]$, and a control $v(t)$ $(t \geqslant t_*)$ admissible for position z_*, such that

$$G^{(u)}_{t^*}(z_*, v(\cdot)) \cap W^{(u)}_{0t^*} = \phi \tag{3.6}$$

$$G^{(u)}_{[t_*,t^*]}(z_*, v(\cdot)) \cap W^{(u)}_{0t^*} = \phi \tag{3.7}$$

From relation (3.6) we obtain the inclusion

$$G^{(u)}_{t^*}(z_*, v(\cdot)) \subset W^{(v)}_{\Sigma} = \bigcup W^{(v)}$$

i.e. every point $z \in G^{(u)}_{t^*}(z_*, v(\cdot))$ belongs to some v-stable bridge $W^{(v)}$. From Lemma 3.2 follows the existence of a finite subcovering of $G^{(u)}_{t^*}(z_*, v(\cdot))$ by some set of the form

$$W^{(v)}_* = \bigcup_{i=1}^{l} W^{(v)}_i \tag{3.8}$$

where $W^{(v)}_i$ $(i = 1, 2, \ldots, l)$ are v-stable bridges. We note that a finite sum of v-stable bridges is a v-stable bridge; therefore the set $W_*^{(v)}$ is a v-stable bridge. Consider the set

$$W^{(v)}_{**} = G_{[t_*,t^*]}(z_*, v(\cdot)) \cup W^{(v)}_{*[t^*,\vartheta]}$$

From relations (3.6), (3.7) we deduce that set $W^{(v)}_{**}$ is a v-stable bridge. But then the inclusion $z_* \in W^{(v)}_{**} \subset W^{(v)}_{\Sigma}$ contradicts the condition $z_* \in W^{(u)}_{0t_*}$ and the definition of set $W^{(u)}_0$. The contradiction obtained proves Theorem 3.2.

The validity of the following alternative ensues right away from Theorems 2.1, 3.1, 3.2.

Theorem 3.3. One of the next two statements is valid for any initial position of the game.

Either $z_0 \in W_0^{(u)}$, and then the encounter problem has a solution which provides a guide-procedure defined for the u-stable bridge $W_0^{(u)}$. In this case an instant $\tau \leqslant \vartheta$ when first

$$(\tau, x(\tau; z_0, W_0^{(u)})) \in M^*$$

exists for any motion $x(t; z_0, W_0^{(u)})$, and

$$(t, x(t; z_0, W_0^{(u)})) \in N^* \quad \text{for} \quad t_0 \leqslant t \leqslant \tau$$

Or $z_0 \in W^{(v)}$, where $W^{(v)}$ is some v-stable bridge, and then the evasion problem has a solution which provides a guide-procedure defined for the bridge $W^{(v)}$. In this case sets G^* and H^* exist, satisfying conditions (3.2), such that the condition

$$(t, x(t; z_0, W^{(v)})) \in G \quad \text{for} \quad t_0 \leqslant t \leqslant \tau^*(x(\cdot))$$

where $\tau^*(x(\cdot))$ is the instant when first

$$(t, x(t; z_0, W^{(v)})) \in H^* \cap \Pi_\vartheta \Pi_\vartheta = \{(t, x) : t = \vartheta, x \in R^n\}$$

is satisfied for every motion $x(t; z_0, W^{(v)})$.

In conclusion we note that the maximal u-stable bridge $W_0^{(u)}$ was here defined formally as the complement of the union of v-stable bridges. However, it is possible to define the bridge $W_0^{(u)}$ differently as the limit of a special sequence of sets defined by a recurrence procedure of program absorption. Such a definition of bridge $W_0^{(u)}$ is analogous to the constructions in [8–10]. When the set $W_0^{(u)}$ is determined by one program absorption operation (see [5, 7], for example), the solving of the encounter problem is simplified and can be reduced to an algorithm implementable on an electronic computer. The position procedure of control with a guide, proposed above, can be realized also when the solution of the encounter problem is known, having been constructed by one of the direct methods under the condition of information discrimination against the player being pursued (see [3, 4], for example).

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